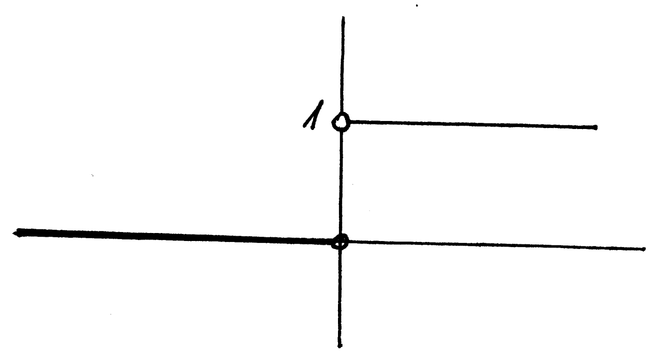
Laplaceova transformacija prekidnih i periodičnih f-ja

F-ja $u(t)$ definisana sa

$$u(t) = \begin{cases} 0, & t<0 \\ 1, & 0<t \end{cases} \qquad (t>0)$$

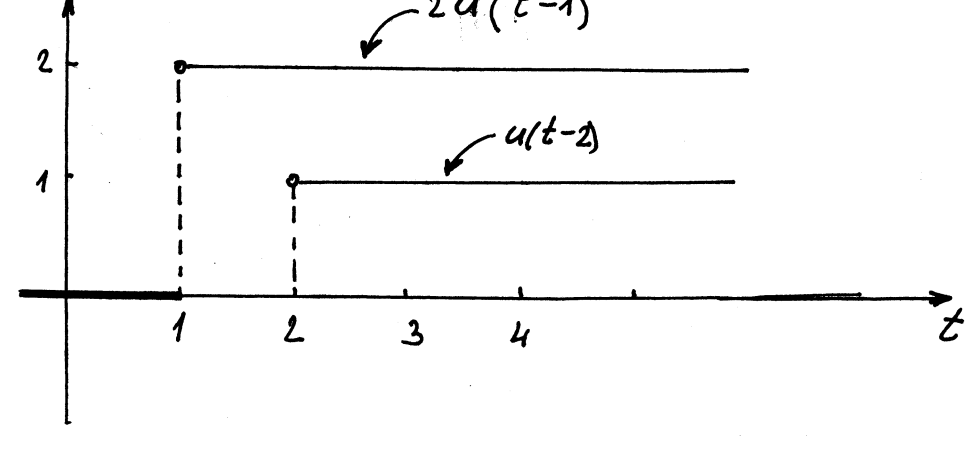
zove se step f-ja ili jedinična f-ja.



Primjetimo da

$$u(t-a) = \begin{cases} 0, & t-a<0 \\ 1, & 0<t-a \end{cases} = \begin{cases} 0, & t<a \\ 1, & a<t \end{cases}$$

$$Mu(t-a) = \begin{cases} 0, & t<a \\ M, & a<t \end{cases}$$



$$\mathcal{L}\{u(t)\}(s) = \int_0^\infty e^{-st}\,dt = \frac{-1}{s} e^{-st}\Big|_0^\infty = \frac{1}{s}, \quad s>0$$

$$\mathcal{L}\{u(t-a)\}(s) = \int_0^\infty e^{-st} u(t-a)\,dt = \int_a^\infty e^{-st}\,dt = \lim_{N\to\infty} \frac{-e^{-st}}{s}\Big|_a^N = \frac{e^{-as}}{s}$$

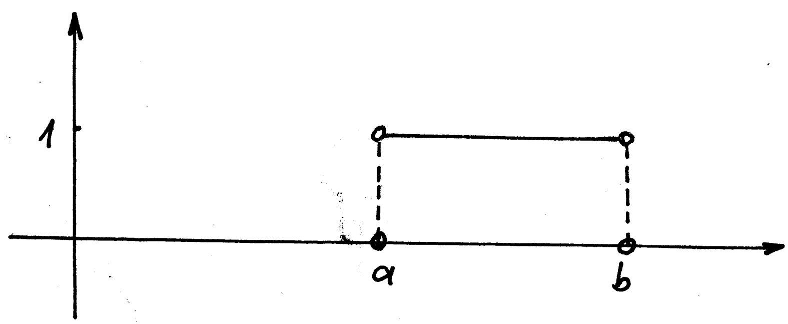
Prema tome

$$\boxed{\mathcal{L}\{u(t)\}(s) = \frac{1}{s}, \quad s>0} \qquad \boxed{\mathcal{L}\{u(t-a)\}(s) = \frac{e^{-as}}{s}, \quad s>0}$$

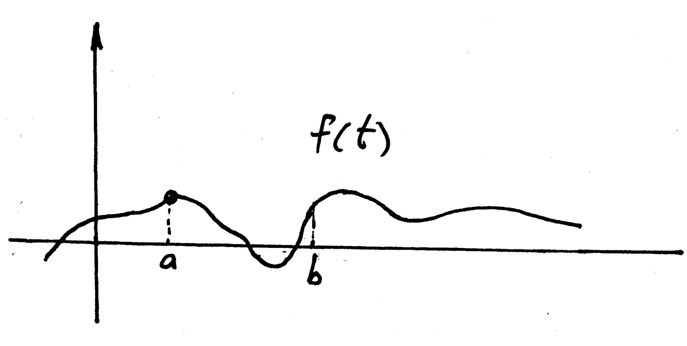
F-ja $\Pi_{a,b}(t)$ definisana sa

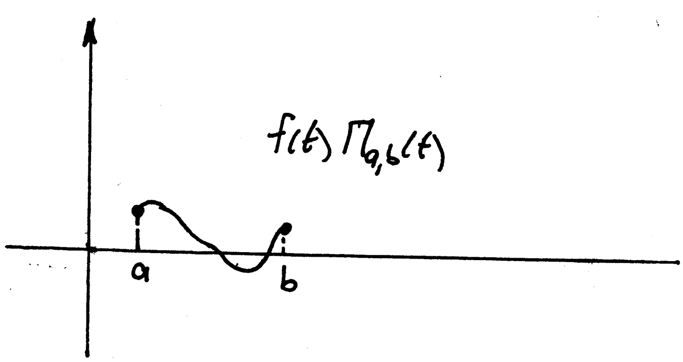
$$\Pi_{a,b}(t) := u(t-a) - u(t-b) = \begin{cases} 0, & t<a \\ 1, & a \le t < b \\ 0, & b<t \end{cases}$$

se naziva gate f-ja ili pravougaona prozor f-ja.



gate f-ja
ili pravougaona
prozor f-ja





Efekat koju pravi f-ja $\Pi_{a,b}(t)$

Datu f-ju

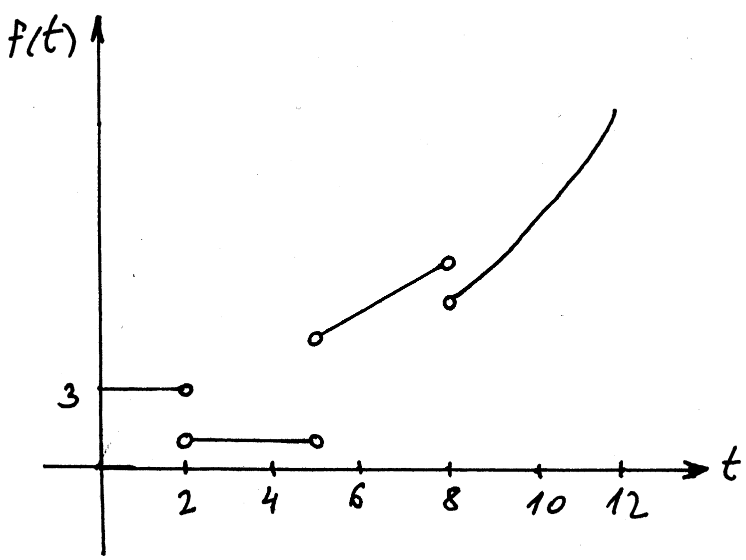
$$f(t)=\begin{cases} 3, & t<2 \\ 1, & 2<t<5 \\ t, & 5<t<8 \\ t^2/10, & 8<t \end{cases}$$

napisati u obliku zbira step i gate f-ja.

Rj.

Primjetimo da nam gate f-ja treba na intervalima $(0,2)$, $(2,5)$ i $(5,8)$ dok ćemo step f-ju upotrebiti za $t>8$.

$$f(t)=3\Pi_{0,2}(t)+1\,\Pi_{2,5}(t)+t\,\Pi_{5,8}(t)+\frac{t^2}{10}u(t-8)$$



Kako je

$$\mathcal{L}\{u(t-a)\}(s) = \frac{e^{-as}}{s}$$

za $a>0$, kažemo da po dijelovima neprekidna f-ja $u(t-a)$ ima inverznu Laplaceovu transformaciju e^{-as}/s i pišemo

$$\mathcal{L}^{-1}\left\{\frac{e^{-as}}{s}\right\}(t) = u(t-a)$$

Za pravougaonu prozor f-ju imamo

$$\mathcal{L}\{\Pi_{a,b}(t)\}(s) = \mathcal{L}\{u(t-a)-u(t-b)\}(s) = \frac{e^{-sa}-e^{-sb}}{s}, \quad 0<a<b$$

Također vrijedi sljedeća osobina za Laplaceovu transformaciju

<u>Translacija po t</u>

Neka je $F(s)=\mathcal{L}\{f\}(s)$ za $s>\alpha\geq 0$ i neka je $a>0$. Tada

$$\boxed{\mathcal{L}\{f(t-a)u(t-a)\}(s) = e^{-as}F(s)}$$

$$\mathcal{L}^{-1}\{e^{-as}F(s)\}(t) = f(t-a)u(t-a).$$

Odavde slijedi da

$$\boxed{\mathcal{L}\{g(t)u(t-a)\}(s) = e^{-as}\mathcal{L}\{g(t+a)\}(s)}$$

Odrediti Laplaceovu transformaciju f-je $t^2 u(t-1)$.

Rj.:

Da bi primjenili sljedeću osobinu Laplaceove transforma.

$$\mathcal{L}\{g(t)u(t-a)\}(s) = e^{-as}\mathcal{L}\{g(t+a)\}(s) \quad \dots (1)$$

uzedemo da je $g(t) = t^2$ i $a = 1$. Tada

$$g(t+a) = g(t+1) = (t+1)^2 = t^2 + 2t + 1$$

pa je

$$\mathcal{L}\{g(t+a)\}(s) = \mathcal{L}\{t^2 + 2t + 1\}(s) = \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}$$

Pa je prema formuli (1)

$$\mathcal{L}\{t^2 u(t-1)\}(s) = e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right)$$

(#) Odrediti $\mathcal{L}\{(\cos t)\, u(t-\pi)\}$.

Rj.: Znamo

$$\underline{\mathcal{L}\{g(t)\,u(t-a)\}(s) = e^{-as}\,\mathcal{L}\{g(t+a)\}(s)} \quad \ldots (1)$$

Ako je $g(t)=\cos t$ i $a=\pi$ imamo

$$g(t+a) = g(t+\pi) = \cos(t+\pi) = \cos t\cos\pi - \sin t\sin\pi = -\cos t$$

$$\mathcal{L}\{g(t+a)\}(s) = -\mathcal{L}\{\cos t\}(s) = -\frac{s}{s^2+1}$$

Sad prema formuli (1)

$$\mathcal{L}\{(\cos t)\,u(t-\pi)\}(s) = -e^{-\pi s}\,\frac{s}{s^2+1}.$$

(#) Odrediti $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\}$ i skicivati grafik.

Rj. Da bi iskoristili osobinu translacije

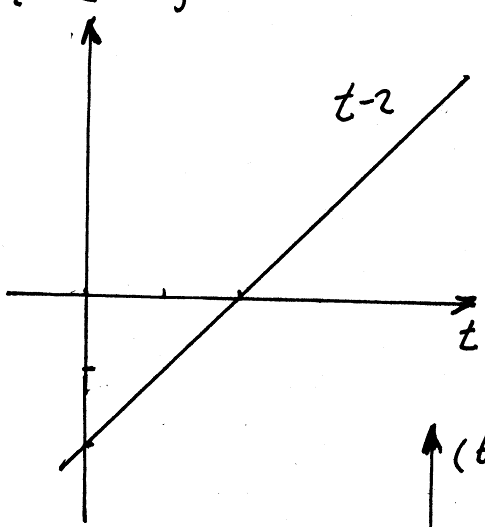
$$\mathcal{L}^{-1}\{e^{-as}F(s)\}(t) = f(t-a)\,u(t-a) \qquad \dots(1)$$

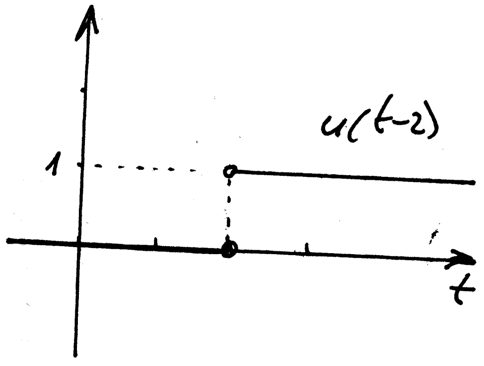
prvo ćemo izraz $\frac{e^{-2s}}{s^2}$ izraziti kao proizvod $e^{-as}F(s)$. Za ovaj zadatak, vidimo da je $e^{-as} = e^{-2s}$ i $F(s) = \frac{1}{s^2}$. Time je $a=2$ pa imamo

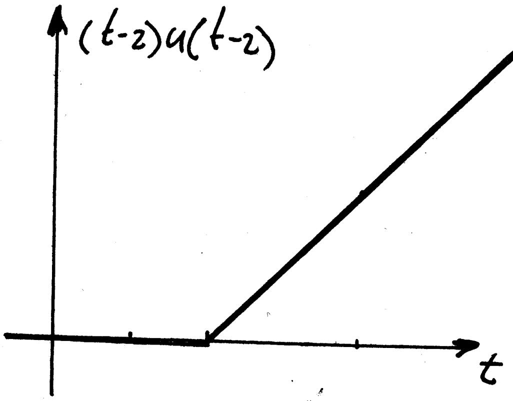
$$F(s) = \frac{1}{s^2} = \mathcal{L}\{f(t)\}(s) \Rightarrow f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}(t) = t$$

Pa ako iskoristimo (1)

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\}(t) = f(t-2)u(t-2) = (t-2)u(t-2)$$







(#) Riješiti diferencijalnu jednačinu

$$y''(t)+4y(t)=g(t); \quad y(0)=0, \quad y'(0)=0$$

gdje je

$$g(t):=\begin{cases} 1, & 0<t<1 \\ -1, & 1<t<2 \\ 0, & 2<t \end{cases}$$

Rj. Označimo sa $F(s)$ Laplasovu transformaciju f-je $y=y(t)$ tj. $F(s):=\mathcal{L}\{y\}(s)$. Tada

$$\mathcal{L}\{y''(t)\}(s)=s^2\mathcal{L}\{y\}(s)-s\cdot y(0)-y'(0)=s^2F(s)$$

Ako f-ju $g(t)$ napišemo u obliku zbira pravougaone prozor f-je $\Pi_{a,b}(t)=u(t-a)-u(t-b)$ dobićemo

$$g(t)=\Pi_{0,1}(t)+(-1)\Pi_{1,2}(t)=u(t)-u(t-1)-\left(u(t-1)-u(t-2)\right)$$
$$=1-2u(t-1)+u(t-2)$$

pa je

$$\mathcal{L}\{g(t)\}(s)=\frac{1}{s}-\frac{2e^{-s}}{s}+\frac{e^{-2s}}{s}$$

Sad ako primjenimo Laplace-ovu transformaciju na obe strane date diferencijalne jednačine $y''(t)+4y(t)=g(t)$ imamo

$$\mathcal{L}\{y''\}(s)+4\mathcal{L}\{y\}(s)=\mathcal{L}\{g\}(s)$$

$$s^2F(s)+4F(s)=\frac{1}{s}-\frac{2e^{-s}}{s}+\frac{e^{-2s}}{s}$$

$$F(s)=\frac{1}{s(s^2+4)}-\frac{2e^{-s}}{s(s^2+4)}+\frac{e^{-2s}}{s(s^2+4)}$$

Sad primjetimo da ako stavimo da je $H(s)=\frac{1}{s(s^2+4)}$ imamo

$$F(s)=H(s)-2e^{-s}H(s)+e^{-2s}H(s)$$

Zašto smo u igru uveli $H(s)$? Želimo da iskoristimo osobinu translacije

$$\underline{\mathcal{L}^{-1}\{e^{-as}F(s)\}(t)=f(t-a)u(t-a)} \qquad \dots (1)$$

$$H(s)=\frac{1}{s(s^2+4)}=\frac{A}{s}+\frac{Bs+D}{s^2+4}=\underset{\dots}{\overset{\text{ZA VJEŽBU}}{}}=\frac{1}{4}\cdot\frac{1}{s}-\frac{1}{4}\left(\frac{s}{s^2+4}\right)$$

$$h(t):=\mathcal{L}^{-1}\{H\}(t)=\frac{1}{4}-\frac{1}{4}\cos 2t$$

Sad ako iskoristimo osobinu translacije (1) imamo

$$\begin{aligned}
y(t)&=\mathcal{L}^{-1}\{H(s)-2e^{-s}H(s)+e^{-2s}H(s)\}(t)\\
&=h(t)-2h(t-1)u(t-1)+h(t-2)u(t-2)\\
&=\left(\frac{1}{4}-\frac{1}{4}\cos 2t\right)-\left(\frac{1}{2}-\frac{1}{2}\cos 2(t-1)\right)u(t-1)\\
&\quad+\left(\frac{1}{4}-\frac{1}{4}\cos 2(t-2)\right)u(t-2)
\end{aligned}$$

Periodične f-je

Za f-ju $f(t)$ kažemo da je periodična perioda T ($\neq 0$) ako je $f(t+T) = f(t)$ za sve t u domenu f-je f.

Transformacija periodične f-je

Ako f-ja f ima period T i po djelovima je neprekidna na intervalu $[0, T]$ tada su Laplace-ove transformacije

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt \quad i \quad F_T(s) = \int_0^T e^{-st} f(t)\, dt$$

povezane sljedećom formulom

$$\boxed{F_T(s) = F(s)\left[1 - e^{-sT}\right]} \quad ili \quad \boxed{F(s) = \frac{F_T(s)}{1 - e^{-sT}}}$$

dokaz za ovo nije težak. Prvo primjetimo da

$$f_T(t) := f(t)\Pi_{0,T}(t) = f(t)\left(u(t) - u(t-T)\right) = \begin{cases} f(t), & 0 < t < T \\ 0, & \text{u suprotnom} \end{cases}$$

pa imamo

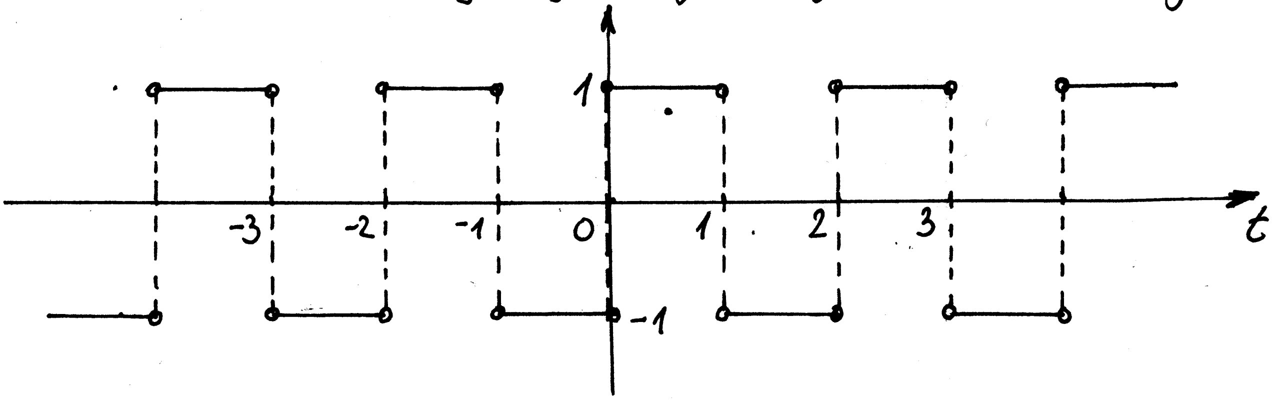
$$F_T(t) = f(t)u(t) - f(t)u(t-T) = f(t)u(t) - f(t-T)u(t-T)$$

sad primjenjujući Laplace-ovu transformaciju na obe strane i iskoristivši osobinu $\mathcal{L}\{f(t-a)u(t-a)\}(s) = e^{-as}F(s)$ dobićemo

$$F_T(s) = F(s) - e^{-sT}F(s)$$

što je ekvivalentno sa zaokruženim formulama.

Odrediti $\mathcal{L}\{f\}$ gdje je f-ja f data grafikom



Data f-ja je poznata pod imenom kvadratna talasna funkcija.

Rj.: Primjetimo da je period ove f-je $T=2$. Tada je

$$f_T(t) = \Pi_{0,1}(t) - \Pi_{1,2}(t)$$

pa prema

$$\mathcal{L}\{\Pi_{a,b}(t)\}(s) = \frac{1}{s}\left(e^{-sa} - e^{-sb}\right)$$

imamo da je $\quad F_T(s) = \frac{1}{s}(1-e^{-s}) - \frac{1}{s}(e^{-s} - e^{-2s}) = \frac{(1-e^{-s})^2}{s}$

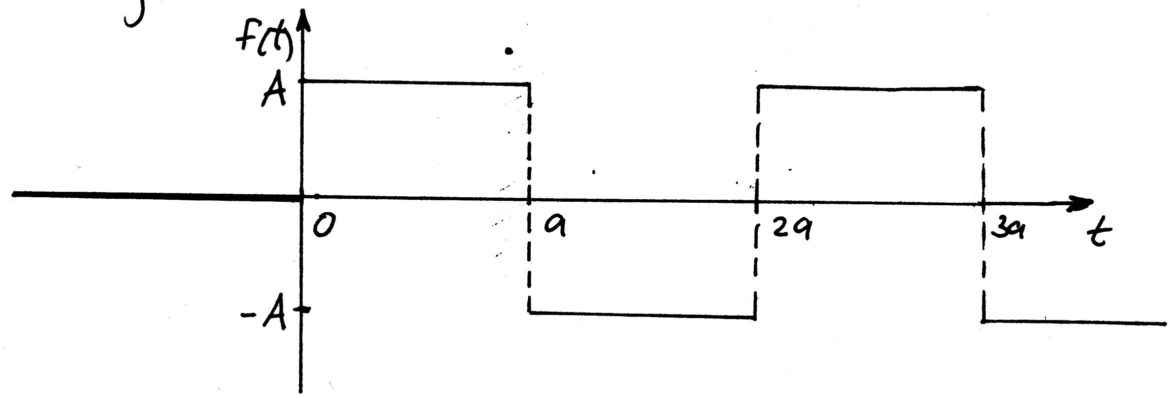
Sad iz osobine transformacije periodične f-je

$$F(s) = \frac{F_T(s)}{1-e^{-sT}}$$

imamo

$$\mathcal{L}\{f\} = \frac{\frac{1}{s}(1-e^{-s})^2}{1-e^{-2s}} = \frac{1-e^{-s}}{(1+e^{-s})\,s}.$$

\# Odrediti Laplaceovu transformaciju f-je f zadanu grafikom



Rj.

Period date f-je je $T=2a$. Imamo da je

$$f_T(t) = A\,\Pi_{0,a}(t) - A\,\Pi_{a,2a}(t)$$

Kako je $\underline{\mathcal{L}\{\Pi_{a,b}(t)\}(s) = \frac{1}{s}(e^{-sa} - e^{-sb})}$, to je

$$\mathcal{L}\{f_T(t)\}(s) = \frac{A}{s}\left(e^{-s\cdot 0} - e^{-sa}\right) - \frac{A}{s}\left(e^{-sa} - e^{-2sa}\right)$$

$$= \frac{A}{s}\underbrace{\left(1 - 2e^{-sa} + \left(e^{-sa}\right)^2\right)}_{(1-e^{-as})^2} = \frac{A}{s}\left(1-e^{as}\right)^2$$

Sad prema osobini transformacije periodične f-je:

$$\underline{F(s) = \frac{F_T(s)}{1-e^{-sT}}}$$

imamo

$$\mathcal{L}\{f\} = \frac{\frac{A}{s}(1-e^{-as})^2}{1-e^{-2sa}} = \frac{A}{s}\cdot\frac{1-e^{-as}}{1+e^{-as}}\; th\left(\frac{as}{2}\right)$$

$$\boxed{\begin{aligned} sh\,x &= \frac{e^{x}-e^{-x}}{2} \\ ch\,x &= \frac{e^{x}+e^{-x}}{2} \end{aligned}}$$

(#) Odrediti $\mathcal{L}\{\sin t\, u(t)\}$.

Rj.

Znamo da

$$\mathcal{L}\{g(t)\, u(t-a)\}(s) = e^{-as}\mathcal{L}\{g(t+a)\}(s)$$

Prema tome za $g(t)=\sin t$, $a=0$ imamo

$$\mathcal{L}\{\sin t\, u(t)\}(s) = e^{0}\mathcal{L}\{\sin t\}(s) = \frac{1}{s^2+1}.$$

(#) Odrediti $\mathcal{L}\{\sin(t-2)\, u(t-2)\}$.

Rj.

Znamo

$$\mathcal{L}\{f(t-a)\, u(t-a)\}(s) = e^{-as}\mathcal{L}\{f\}(s)$$

za $f(t-a)=\sin(t-a)$ i $a=2$ imamo

$$\mathcal{L}\{\sin(t-2)\, u(t-2)\}(s) = e^{-2s}\mathcal{L}\{\sin t\}(s) = \frac{e^{-2s}}{s^2+1}$$

Odrediti Laplace-ove transformacije sljedećih f-ja

(a) $(t-2)^2 u(t)$ (c) $(t-2)^2 e^{-t}$

(b) $(t-2)^2 u(t-2)$ (d) $(t-2)^2 u(t-2)\, e^{-t}$

Rj. U rješenju ćemo koristiti sljedeće dvije formule

$\mathcal{L}\{f(t-a)u(t-a)\}(s) = e^{-as}\mathcal{L}\{f\}(s)$ i $\mathcal{L}\{g(t)u(t-a)\}(s) = e^{-as}\mathcal{L}\{g(t+a)\}(s)$

(a) $(t-2)^2 = t^2 - 4t + 4$

$$\mathcal{L}\{(t-2)^2 u(t)\} = \mathcal{L}\{t^2 u(t)\}(s) - 4\mathcal{L}\{t u(t)\}(s) + 4\mathcal{L}\{u(t)\}(s)$$

$$= \frac{2}{s^3} - \frac{4}{s^2} + \frac{4}{s}$$

(b) Kako je $\mathcal{L}\{t^2\}(s) = \frac{2}{s^3}$ to je $\mathcal{L}\{(t-2)^2 u(t-2)\} = e^{-2s}\frac{2}{s^3}$

(c) Prema osobini translacije po s $\mathcal{L}\{e^{at} f(t)\}(s) = F(s-a)$

i dijela pod (a) imamo

$$\mathcal{L}\{(t-2)^2 e^{-t}\} = \frac{2}{(s+1)^3} - \frac{4}{(s+1)^2} + \frac{4}{s+1}$$

(d) Možemo iskoristiti osobinu translacije po s i dio pod (b)

$$\mathcal{L}\{(t-2)^2 u(t-2) e^{-t}\}(s) = e^{-2(s+1)}\frac{2}{(s+1)^3}$$

(#) Odrediti Laplace-ove transformacije f-ja

(a) $f(t) = (t-3)^4 e^{-2t} u(t-3)$

(b) $g(t) = (t-3)^4 e^{-2t} u(t)$.

Rj.

(a) $$\mathcal{L}\{t^4\} = \frac{4!}{s^5} = \frac{24}{s^5}$$

$$\mathcal{L}\{(t-3)^4 u(t-3)\}(s) = \frac{24}{s^5} \cdot e^{-3s}$$

$$\mathcal{L}\{e^{-2t}(t-3)^4 u(t-3)\}(s) = \frac{24}{(s+2)^5} e^{-3(s+2)}$$

(b)

$$\mathcal{L}\{(t-3)^4 e^{-2t} u(t)\} =$$

$$= (t^4 - 12t^3 + 54t^2 - 108t + 81) e^{-2t} u(t)$$

$$= \frac{24}{(s+2)^5} - 12 \frac{6}{(s+2)^4} + 54 \frac{2}{(s+2)^3} - 108 \frac{1}{(s+2)^2} + 81 \frac{1}{s+2}$$

(#) Odrediti Laplaceovu transformaciju f-je

$$f(t) = \begin{cases} 3, & 0<t<2 \\ -1, & 2<t<4 \\ 0, & t>4 \end{cases}$$

Rj: U rješenju koristimo prikaz pomoću gate f-je

$$f(t) = 3\Pi_{0,2}(t) - \Pi_{2,4}(t) = 3(u(t)-u(t-2)) - (u(t-2)-u(t-4)) =$$

$$= 3u(t) - 4u(t-2) + u(t-4)$$

$$\mathcal{L}\{f(t)\}(s) = 3\cdot\frac{1}{s} - 4\cdot\frac{e^{-2s}}{s} + \frac{e^{-4s}}{s} = \frac{3-4e^{-2s}+e^{-4s}}{s}$$

(#) Odrediti Laplasovu transformaciju f-je,

$$f(t) = \begin{cases} 2, & 0<t<1 \\ 3t, & 1<t<2 \\ 4e^{t}, & 2<t \end{cases}$$

Rj:

$$f(t) = 2\Pi_{0,1}(t) + 3t\,\Pi_{1,2}(t) + 4e^{t}u(t-2) =$$

$$= 2(u(t)-u(t-1)) + 3t(u(t-1)-u(t-2)) + 4e^{t}u(t-2) =$$

$$= 2u(t) + (3t-2)u(t-1) + (4e^{t}-3t)u(t-2)$$

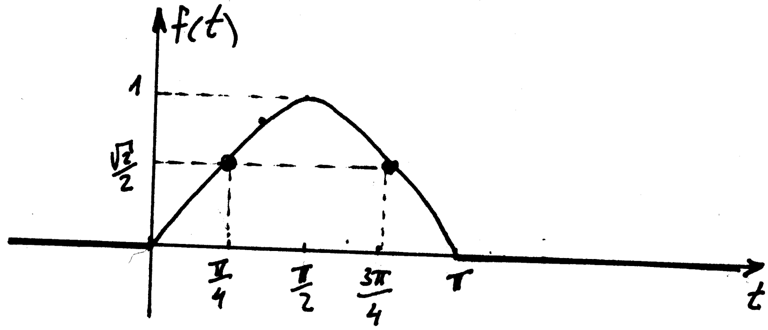
$$= 2u(t) + [3(t-1)+1]u(t-1) + [4e^{2}e^{t-2} - 3(t-2) - 6]u(t-2)$$

Kako je $\mathcal{L}\{2\} = \frac{2}{s}$; $\mathcal{L}\{3t+1\}(s) = \frac{3}{s^2} + \frac{1}{s}$; $\mathcal{L}\{4e^{2}e^{t} - 3t - 6\}(s) = 4e^{2}\frac{1}{s-1} + \frac{3}{s^2} - \frac{6}{s}$

imamo $\mathcal{L}\{f(t)\}(s) = e^{-0\cdot s}\frac{2}{s} + e^{-1\cdot s}\left(\frac{3}{s^2}+\frac{1}{s}\right) + e^{-2s}\left(\frac{4e^2}{s-1} - \frac{3}{s^2} - \frac{6}{s}\right) =$

$$= \frac{2}{s} + \frac{s+3}{s^2}e^{-s} + \left(\frac{4e^2}{s-1} - 3\,\frac{2s+1}{s^2}\right)e^{-2s}.$$

(#) Odrediti Laplaceovu transformaciju f-j zadanu grafikom:



Rj. Primjetimo da datu f-ju možemo napisati u obliku

$$f(t)=\begin{cases}\sin t, & 0\leq t\leq \pi\\ 0, & \text{inače}\end{cases}$$

ili sa

$$f(t)=\sin t\cdot \Pi_{0,\pi}(t)=\sin t\,[u(t)-u(t-\pi)]$$

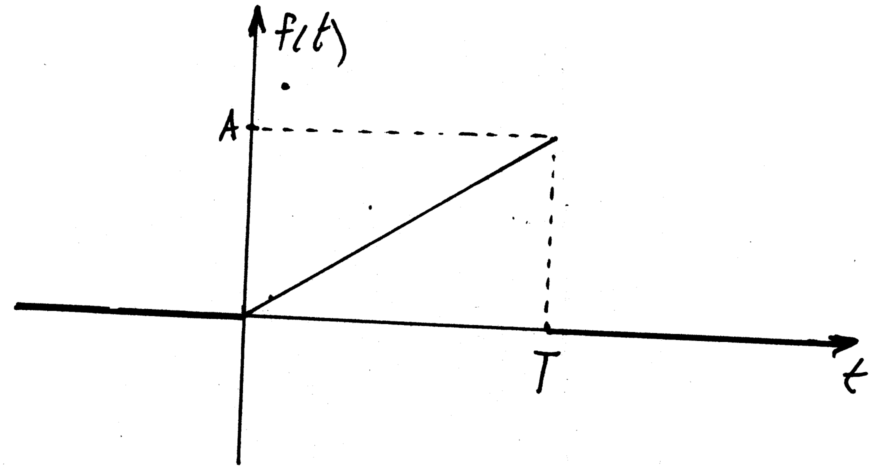
$$=\sin t\, u(t)-\sin t\, u(t-\pi)$$

Kako je $\sin(t-\pi)=\sin t\cos\pi+\sin\pi\cos t=-\sin t$ bo je

$$f(t)=\sin t\, u(t)+\sin(t-\pi)u(t-\pi)$$

$$\mathcal{L}\{f(t)\}(s)=\frac{1}{s^2+1}+\frac{e^{-\pi s}}{s^2+1}$$

Odrediti Laplasovu transformaciju f-je zadanu grafikom:



Rj.: Data je sljedeća f-ja

$$f(t) = \begin{cases} \frac{A}{T} t, & 0 < t < T \\ 0, & \text{inače} \end{cases}$$

F-ju možemo napisati u obliku

$$f(t) = \frac{A}{T} t \, \Pi_{0,T}(t) = \frac{A}{T}\left(u(t) - u(t-T)\right)$$

Da bismo odredili njezinu transformaciju moramo je dovesti u odgovarajući oblik

$$f(t) = \frac{A}{T} t\, u(t) - \frac{A}{T}(t - T + T)\, u(t-T)$$

$$= \frac{A}{T} t u(t) - \frac{A}{T}(t-T)\, u(t-T) - A u(t-T)$$

$$\mathcal{L}\{f(t)\}(s) = \frac{A}{T} \cdot \frac{1}{s^2} - \frac{A}{T} \cdot \frac{e^{-Ts}}{s^2} - \frac{A e^{-Ts}}{s}$$

Odrediti $\mathcal{L}^{-1}\left\{ \dfrac{e^{-s\pi}(1+e^{-s\pi})}{s^2+1} \right\}$.

Rj. Napišimo orginal u obliku

$$F(s) = \frac{1}{s^2+1}e^{-\pi s} + \frac{1}{s^2+1}e^{-2\pi s}$$

Sad imamo

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}(t) = \sin t$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}e^{-\pi s}\right\}(t) = \sin(t-\pi)u(t-\pi)$$

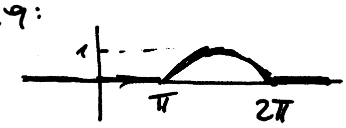
$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}e^{-2\pi s}\right\}(t) = \sin(t-2\pi)u(t-2\pi)$$

Prema tome

$$f(t) = \underbrace{\sin(t-\pi)}_{=-\sin t}u(t-\pi) + \underbrace{\sin(t-2\pi)}_{=\sin t}u(t-2\pi) =$$

$$= -\sin u(t-\pi) + \sin t\, u(t-2\pi) = -\sin t\left(u(t-\pi) - u(t-2\pi)\right)$$

$$= -\sin t\, \Pi_{\pi,2\pi}(t) = \begin{cases} -\sin t, & \pi < t < 2\pi \\ 0, & \text{inače} \end{cases}$$

(za vježbu skicirati grafik f-je $f(t)$, uputa: )

Zadaci za vježbu

(1.) Skicirati graf datih f-ja i odrediti njihove Laplaceove transformacije

(a) $(t-1)^2 u(t-1)$

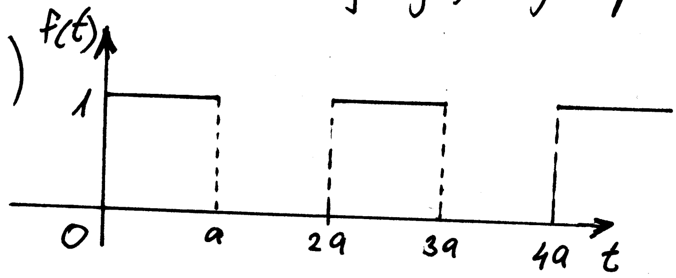
(b) $t^2 u(t-2)$

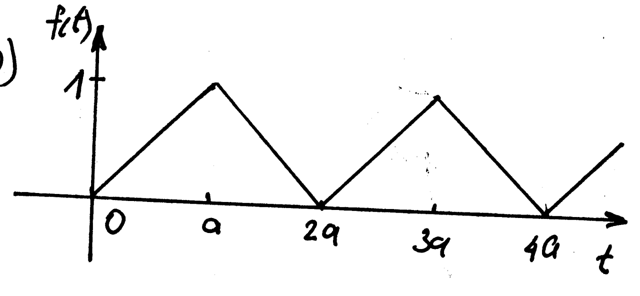
(2.) Odrediti inverznu Laplaceovu transformaciju datih f-ja

(a) $\dfrac{e^{-2s}}{s-1}$ (b) $\dfrac{e^{-2s}-3e^{-4s}}{s+2}$ (c) $\dfrac{se^{-3s}}{s^2+4s+5}$

(d) $\dfrac{e^{-3s}(s-5)}{(s+1)(s+2)}$ (e) $\dfrac{e^{-s}(3s^2-s+2)}{(s-1)(s^2+1)}$

(3.) Odrediti $\mathcal{L}\{f\}$ gdje je periodična f-ja f opisana grafikom

(a) 

(b) 

(4.) Riješiti diferencijalnu jednačinu sa datim uslovom. Skicirati grafik rješenja

(a) $y''+y=u(t-3)$, $y(0)=0$, $y'(0)=1$ (b) $y''+y=t-(t-4)u(t-2)$, $y(0)=0$, $y'(0)=1$

(5.) Riješiti date diferencijalne jednačine koristeći metodu Laplaceove transformacije

(a) $y''+2y'+2y=u(t-2\pi)-u(t-4\pi)$, $y(0)=1$, $y'(0)=1$;

(b) $z''+3z'+2z=e^{-3t}u(t-2)$, $z(0)=2$, $z'(0)=-3$;

(c) $y''+4y=g(t)$; $y(0)=1$, $y'(0)=3$, gdje je $g(t)=\begin{cases}\sin t, & 0\le t\le 2\pi\\ 0, & 2\pi<t\end{cases}$

(d) $y'' + 5y' + 6y = g(t)$, $y(0) = 0$, $y'(0) = 2$, gdje je

$$g(t) = \begin{cases} 0, & 0 \le t < 1 \\ t, & 1 < t < 5 \\ 1, & 5 < t \end{cases}$$

Odgovori:

1.
(a) $\dfrac{2e^{-s}}{s^3}$ (b) $\dfrac{e^{-2s}(4s^2+4s+2)}{s^3}$

2.
(a) $e^{t-2}u(t-2)$ (b) $e^{-2(t-2)}u(t-2) - 3e^{-2(t-4)}u(t-4)$

(c) $e^{-2(t-3)}[\cos(t-3) - 2\sin(t-3)]u(t-3)$

(d) $(7e^{6-2t} - 6e^{3-t})u(t-3)$

3.
(a) $\dfrac{1}{s(1+e^{-as})}$ (b) $\dfrac{1-e^{-as}}{as^2(1+e^{-as})}$

4.
(a) $\sin t + [1 - \cos(t-3)]u(t-3)$

(b) $t + [4 - t + \sin(t-2) - 2\cos(t-2)]u(t-2)$

5.
(a) $e^{-t}\cos t + 2e^{-t}\sin t + \frac{1}{2}[1 - e^{2\pi-t}(\cos t + \sin t)]u(t-2\pi)$
$- \frac{1}{2}[1 - e^{4\pi-t}(\cos t + \sin t)]u(t-4\pi)$

(b) $e^{-t} + e^{-2t} + \frac{1}{2}[e^{-3t} - 2e^{-2(t+1)} + e^{-(t+4)}]u(t-2)$

(c) $\cos 2t + \frac{1}{3}[1 - u(t-2\pi)]\sin t + \frac{1}{6}[8 + u(t-2\pi)]\sin 2t$

(d) $2e^{-2t} - 2e^{-3t} + \left[\frac{1}{36} + \frac{1}{6}(t-1) - \frac{1}{4}e^{-2(t-1)} + \frac{2}{9}e^{-3(t-1)}\right]u(t-1) - \left[\frac{13}{36} + \frac{1}{6}(t-5) - \frac{7}{4}e^{-2(t-5)} + \frac{11}{9}e^{-3(t-5)}\right]u(t-5)$